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Journal of Geometry and Physics 20 (1996) 273–296

JOURNAL OF GEOMETRY AND PHYSICS

Mirror symmetry of elliptic curves and Ising model

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Received 2 August 1995; revised 21 November 1995

Abstract

We study the differential equations governing mirror symmetry of elliptic curves, and obtain a characterization of the ODEs which give rise to the integral **q**-expansion of mirror maps. Through theta function representation of the defining equation, we express the mirror correspondence in terms of theta constants. By investigating the elliptic curves in X_9-family, the identification of the Landau–Ginzburg potential with the spectral curve of Ising model is obtained. Through the Jacobi elliptic function parametrization of Boltzmann weights in the statistical model, an exact Jacobi form-like formula of mirror map is described.

Subj. Class.: Quantum field theory
1991 MSC: 14H52, 14K25
PACS: 05.50
Keywords: Mirror maps; Elliptic curves; Ising model

1. Introduction

Recent progress in physicists' construction [6,17] of the "number" of rational curves of arbitrary degree on a large class of Calabi–Yau spaces has stimulated efforts to find a mathematical understanding of this remarkable "counting" principle. As is known the main ingredient of practically all examples is to express the "counting" function, called the mirror map, in terms of solutions of a generalized hypergeometric system. As a physical theory, it is the N = 2 supersymmetry (SUSY) two-dimensional Landau–Ginzburg (LG) models to describe the mirror symmetry of σ-models on Kähler manifolds with vanishing first Chern class (for the basic notion of mirror symmetry, we refer readers to [17]). This novel principle gives also counting functions on other $c_1 = 0$ algebraic manifolds of an arbitrary (complex)

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SSDI 0393-0440(95)00060-7

Table 1

	Constraint	Differential operator	$1728J(z)$
P_8	$x_1^3+x_2^3+x_3^3-z^{-1/3}x_1x_2x_3=0$ in $\mathbb{P}^2_{(1,1,1)}$	$\Theta^2-3z(3\Theta+2)(3\Theta+1)$	$(1+216z)^3/z(1-27z)^3$
X_9	$x_1^4+x_2^4+x_3^2-z^{-1/4}x_1x_2x_3=0$ in $\mathbb{P}^2_{(1,1,2)}$	$\Theta^2-4z(4\Theta+3)(4\Theta+1)$	$(1+192z)^3/z(1-64z)^2$
J_{10}	$x_1^6+x_2^3+x_3^2-z^{-1/6}x_1x_2x_3=0$ in $\mathbb{P}^2_{(1,2,3)}$	$\Theta^2-12z(6\Theta+5)(6\Theta+1)$	$1/z(1-432z)$

dimension. In the elliptic curve case, there are three ways of realizing them as hypersurfaces in weighted projective 2-space, and the moduli parameter is always connected to the classical J-function by an algebraic relation [8,9] presented in Table 1. Here the differential operator describes the Picard–Fuchs equation for the family, and $\Theta := z(\partial/\partial z)$. With the variable $\mathbf{t}$ obtained by a ratio of fundamental solutions of the differential equation near $z=0$, the mirror map yields the following numerical expansion of $\mathbf{q}$ ($:= e^{2\pi i\mathbf{t}}$) for the parameter z:

$$\begin{aligned} P_8:\ & z(\mathbf{q})=\mathbf{q}-15\mathbf{q}^2+171\mathbf{q}^3-1679\mathbf{q}^4+15\,054\mathbf{q}^5-126\,981\mathbf{q}^6+\cdots, \\ X_9:\ & z(\mathbf{q})=\mathbf{q}-40\mathbf{q}^2+1324\mathbf{q}^3-39\,872\mathbf{q}^4+1\,136\,334\mathbf{q}^5 \\ & \qquad -31\,239\,904\mathbf{q}^6+\cdots, \\ J_{10}:\ & z(\mathbf{q})=\mathbf{q}-312\mathbf{q}^2+87\,084\mathbf{q}^3-23\,067\,968\mathbf{q}^4+5\,930\,898\,126\mathbf{q}^5+\cdots. \end{aligned} \tag{1}$$

Note that the "counting" numbers in these expansions are all integers. For a general Calabi–Yau hypersurface family in a weighted projective space, one also produces a "counting" function of such kind. However the mathematical reason for the arithmetical nature of "counting" functions is poorly understood, but a fundamental understanding of this counting principle should be important to further mathematical development of mirror symmetry. In [8], the generalized Schwarzian equations were derived for mirror maps of one-modulus cases as one effort towards this direction. The starting point of the present work is to clarify the role of differential equations in the integral property of counting function $z(\mathbf{q})$. We find a characterization of the equations appeared in Table 1 by their qualitative relations with J-function. For the precise statement of the result, see Theorem 1. On the other hand, the numerical evidence has also suggested $z(\mathbf{q})$ might possess a certain structure like modular functions. To the author's knowledge, not much is known about the exact modular form-like expression of $z(\mathbf{q})$, even on elliptic curve cases. In this paper, we have obtained the elliptic theta function parametrization of constraints, i.e. LG superpotentials, in Table 1, and also the exact formula of $z(\mathbf{q})$ in terms of theta constants. The key ingredient is the observation of the connection between discrete symmetries encoded in constraints and their hidden theta function (projective) representations. Our purpose here is to analyse extensively the discrete symmetries appeared in Table 1 and to determine the theta function parametrization of the superpotential for each case, which allows one to obtain the exact formula of the moduli parameter. One main contribution of the present work is that we have connected X_9-family with Ising model, a standard physical theory which has been served as a basis to provide a simple two-dimensional statistical model. Here the Jacobi elliptic parametrization of Boltzmann weights in Ising model is used for the derivation of theta function representation of the X_9-potential, and the Jacobi form expression of temperature-like parameter of Ising family leads to a closed form of $z(\mathbf{q})$ for X_9 in (1) in terms of theta constants. With this novel phenomena, it becomes increasingly interesting in the interplay of geometry of $c_1 = 0$ Kähler manifolds with other two-dimensional solvable statistical models. As is well known, theta function parametrizations have provided a powerful tool in two-dimensional lattice models to obtain quantities of physical interest [3,15]. In recent years, there has been considerable progress in the study of chiral Potts N-state models [1,5] as a generalization of Ising model. The Boltzmann weights of the chiral Potts models lie on hyperelliptic curves with a large number of discrete symmetries, and their theta function parametrizations are known [4,13]. The question that we address for future investigation is to establish a connection between this hyperelliptic function parametrization with mirror maps of Calabi–Yau spaces. A resolution might point towards some future structure, yet to be explained.

The following is a summary of the contents of this article: In Section 2, we recall some basic facts on elliptic theta functions and Heisenberg group representation which will be needed for the discussion of this paper. In Section 3, we study the Schwarzian equations satisfied by mirror maps, which are derived from a special type of Fuchsian differential equations [8,10]. We characterize the differential operators in Table 1, which are solely governed by the integral property of the $\mathbf{q}$-expansion of the Schwarz triangle function and its qualitative relation with J-function. Also we indicate the J_{10}-family as an equivalent version of Weierstrass form of elliptic curves. In Section 4, the elliptic theta function parametrization of P_8-family is derived, so is the expression of $z(\mathbf{q})$ in terms of theta constants. Based on the identification of symmetries of the defining equation with the finite Heisenberg group of degree 3, the standard theta function representation of the group gives rise to the parametrization of P_8-potential. In Section 5, we give a brief review on elliptic curve theory related to Boltzmann weights of Ising model, which will be relevant to our discussion. Primary focus is on its Jacobi elliptic function parametrization. With this parametrization, by examining the relation between X_9-potential and Ising model, we derive the Jacobi elliptic function representation of elliptic X_9-family in Section 6, and also the exact formula for the moduli parameter $z(\mathbf{q})$. After carrying out the mathematical results of this paper, finally in Section 7 we will mention a comparison of some essential structures of two physical theories: N $= 2$ SUSY LG theory and exactly solvable statistical model, whose geometry is presented in Calabi–Yau spaces and hyperelliptic curves of chiral Potts models, respectively.

In this paper, we use the following notations: $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ the complex upper-half plane, $\mathbb{E}_{a,b}$ is the one-dimensional torus $\mathbb{C}/(\mathbb{Z}a + \mathbb{Z}b)$ for two $\mathbb{R}$-independent complex numbers a, b, and $\mathbb{E}_{a,b}(d)$ the d-torsion of $\mathbb{E}_{a,b}$ for a positive integer d.

2. Preliminary

Here we recall the definitions of Heisenberg group and theta functions, and list some of their basic properties that will be used in the context of this paper. For the details , we refer the readers to some standard text books on theta functions, e.g. [11].

Definition 1.

(i) $\mathbb{G} = \mathbb{C}_1^* \times \mathbb{R} \times \mathbb{R}$ ($\mathbb{C}_1^* = \{\alpha \in \mathbb{C}^* \mid |\alpha| = 1\}$): the (three-dimensional) Heisenberg group with the group law, $(\alpha, \delta, \mu) \cdot (\alpha', \delta', \mu') = (\alpha\alpha' e^{2\pi i \mu\delta'}, \delta + \delta', \mu + \mu')$. $\Lambda(p, q)$ is the subgroup of $\mathbb{G}$ generated by $(1, p, 0)$ and $(1, 0, q)$ for non-zero rational numbers p, q.

(ii) $\mathbb{G}_d = \nu_d \times \mathbb{Z}_d \times \mathbb{Z}_d$ ($\nu_d = \{\alpha \in \mathbb{C}^* \mid \alpha^d = 1\}$): the finite Heisenberg group of degree d with the group law, $(\alpha, \overline{\delta}, \overline{\mu}) \cdot (\alpha', \overline{\delta'}, \overline{\mu'}) = (\alpha\alpha' e^{2\pi i \mu\delta'/d}, \overline{\delta + \delta'}, \overline{\mu + \mu'})$.

(iii) $\tilde{\mathbb{G}}_d = \mathbb{G}_d \cdot \mathbb{Z}_2$: the extended degree d Heisenberg group which is the semidirect product of $\mathbb{G}_d$ and $\mathbb{Z}_2$ with $\mathbb{G}_d$ normal in $\tilde{\mathbb{G}}_d$, where the conjugate action on $\mathbb{G}_d$ by the non-trivial element of $\mathbb{Z}_2$ is the map, $(\alpha, \overline{\delta}, \overline{\mu}) \longmapsto (\alpha, -\overline{\delta}, -\overline{\mu})$ for $(\alpha, \overline{\delta}, \overline{\mu}) \in \mathbb{G}_d$.

(iv) The canonical representation of $\tilde{\mathbb{G}}_d$ is the d-dimensional irreducible representation of $\tilde{\mathbb{G}}_d$ where group elements act on a basis $\{e_k\}_{k=0}^{d-1}$ of the vector space by

$$\begin{aligned} ((\alpha, 0, 0) \times 0)e_k &= \alpha e_k, & ((1, 0, 0) \times \overline{1})e_k &= e_{d-k} \quad (e_d := e_0), \\ ((1, \overline{1}, 0) \times 0)e_k &= e_{k+1}, & ((1, 0, \overline{1}) \times 0)e_k &= e^{2\pi i k/d} e_k. \end{aligned} \tag{2}$$

It is easy to see that the following groups are isomorphic:

$$\begin{aligned} &\Lambda(1/d, 1)/\Lambda(1, d) \simeq \mathbb{G}_d, \\ &(1, 1/d, 0) + \Lambda(1, d) \longmapsto (1, \bar{1}, 0), \\ &(1, 0, 1) + \Lambda(1, d) \longmapsto (1, 0, \bar{1}). \end{aligned} \tag{3}$$

For $\delta, \mu \in \mathbb{R}$, $\tau \in \mathbb{H}$ and an entire function f on $\mathbb{C}$, one defines the functions $S_\mu f$ and $T_\delta(\tau) f$ by

$$(S_\mu f)(z) = f(z + \mu), \quad (T_\delta(\tau) f)(z) = q^{\delta^2} e^{2\pi i \delta z} f(z + \delta\tau) \qquad \text{for } z \in \mathbb{C}.$$

Then S_μ and T_δ ($= T_\delta(\tau)$) act on the space of entire functions with the relations

$$S_\mu S_{\mu'} = S_{\mu+\mu'}, \qquad T_\delta T_{\delta'} = T_{\delta+\delta'}, \qquad S_\mu T_\delta = e^{2\pi i \mu\delta} T_\delta S_\mu \tag{4}$$

and they generate a representation of the Heisenberg group $\mathbb{G}$ by $(\alpha, \delta, \mu) f := \alpha T_\delta(\tau) S_\mu f$ for $(\alpha, \delta, \mu) \in \mathbb{G}$ and f an entire function. For $\tau \in \mathbb{H}$, the theta function $\vartheta(z, \tau)$ of $\mathbb{E}_{\tau,1}$ and the theta function $\vartheta\left[\begin{smallmatrix}\delta\\ \mu\end{smallmatrix}\right](z, \tau)$ with characteristics $\delta, \mu \in \mathbb{R}$ are defined by

$$\vartheta(z) \ \ (= \vartheta(z, \tau)) := \sum_{m \in \mathbb{Z}} (T_n 1)(z) = \sum_{m=-\infty}^{\infty} q^{m^2} e^{2\pi i m z},$$

$$\vartheta \begin{bmatrix} \delta \\ \mu \end{bmatrix} (z) \left(= \vartheta \begin{bmatrix} \delta \\ \mu \end{bmatrix} (z, \tau)\right) := S_\mu T_\delta \vartheta(z) = q^{\delta^2} e^{2\pi i \delta(z+\mu)} \vartheta(z + \delta\tau + \mu), \ z \in \mathbb{C},$$

where 1 is the constant function with value one and $q := e^{\pi i \tau}$. (Note that the variable $\mathbf{q}$ in Section 1 is related to q by $\mathbf{q} = q^2$.) Then $\vartheta(z, \tau)$ is the unique entire function invariant under $\Lambda(1, 1)$, and we have

$$T_\alpha \vartheta \begin{bmatrix} \delta \\ \mu \end{bmatrix} = e^{-2\pi i \alpha\mu} \vartheta \begin{bmatrix} \delta + \alpha \\ \mu \end{bmatrix}, \qquad S_\beta \vartheta \begin{bmatrix} \delta \\ \mu \end{bmatrix} = \vartheta \begin{bmatrix} \delta \\ \mu + \beta \end{bmatrix}. \tag{5}$$

The theta function has the representation of infinite product:

$$\vartheta(z,\tau)=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau)=\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2\pi z+q^{4n-2})(1-q^{2n}),$$

and the following quasi-periodicity and zero relation hold for $\vartheta\left[\begin{smallmatrix}\delta\\ \mu\end{smallmatrix}\right]$:

$$\begin{aligned}
&\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z+1,\tau)=e^{2\pi i\delta}\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z,\tau),\\
&\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z+\tau,\tau)=q^{-1}e^{-2\pi i(z+\mu)}\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z,\tau),\\
&\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z,\tau)=0\iff z\equiv(\tfrac{1}{2}-\delta)\tau+(\tfrac{1}{2}-\mu)\,(\mathrm{mod}\,\mathbb{Z}\tau+\mathbb{Z}).
\end{aligned}\tag{6}$$

We have

$$\begin{aligned}
&\vartheta\begin{bmatrix}\delta'+\delta\\ \mu'+\mu\end{bmatrix}(z,\tau)=q^{\delta^2}e^{2\pi i\delta(z+\mu+\mu')}\vartheta\begin{bmatrix}\delta'\\ \mu'\end{bmatrix}(z+\delta\tau+\mu,\tau),\\
&\vartheta\begin{bmatrix}\delta+1\\ \mu\end{bmatrix}(z,\tau)=\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z,\tau),\qquad \vartheta\begin{bmatrix}\delta\\ \mu+1\end{bmatrix}(z,\tau)=e^{2\pi i\delta}\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(z,\tau).
\end{aligned}\tag{7}$$

Hence

$$\begin{aligned}
&\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(-z,\tau)=\vartheta\begin{bmatrix}1-\delta\\ -\mu\end{bmatrix}(z,\tau)=e^{-2\pi i\delta}\vartheta\begin{bmatrix}1-\delta\\ 1-\mu\end{bmatrix}(z,\tau),\\
&\vartheta\begin{bmatrix}1-\delta\\ 1-\mu\end{bmatrix}(\tfrac{1}{2}(\tau+1),\tau)=-e^{2\pi i\mu}\vartheta\begin{bmatrix}\delta\\ \mu\end{bmatrix}(\tfrac{1}{2}(\tau+1),\tau).
\end{aligned}\tag{8}$$

The infinite product representation of theta functions with half-integer characteristics are given by:

$$\begin{aligned}
\vartheta_1(z,\tau)&:=\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z,\tau)=2q_0q^{1/4}\sin\pi z\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2\pi z+q^{4n}),\\
\vartheta_2(z,\tau)&:=\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(z,\tau)=2q_0q^{1/4}\cos\pi z\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2\pi z+q^{4n}),\\
\vartheta_3(z,\tau)&:=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau)=q_0\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2\pi z+q^{4n-2}),\\
\vartheta_4(z,\tau)&:=\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(z,\tau)=q_0\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2\pi z+q^{4n-2}),
\end{aligned}\tag{9}$$

where $q_0:=\prod_{n=1}^{\infty}(1-q^{2n})$. The odd function ϑ_1 and even functions $\vartheta_2,\vartheta_3,\vartheta_4$ of variable z (for the same modular τ) satisfy the square relations:

$$\begin{aligned}\vartheta_1^2(z)\vartheta_3^2(0) &= \vartheta_4^2(z)\vartheta_2^2(0) - \vartheta_2^2(z)\vartheta_4^2(0),\\ \vartheta_1^2(z)\vartheta_4^2(0) &= \vartheta_3^2(z)\vartheta_2^2(0) - \vartheta_2^2(z)\vartheta_3^2(0),\\ \vartheta_4^2(z)\vartheta_4^2(0) &= \vartheta_3^2(z)\vartheta_3^2(0) - \vartheta_2^2(z)\vartheta_2^2(0),\\ \vartheta_1^2(z)\vartheta_2^2(0) &= \vartheta_4^2(z)\vartheta_3^2(0) - \vartheta_3^2(z)\vartheta_4^2(0),\end{aligned} \tag{10}$$

hence the identity of their zero argument:

$$\vartheta_2^4(0,\tau) + \vartheta_4^4(0,\tau) = \vartheta_3^4(0,\tau). \tag{11}$$

For a positive integer d, let $\mathrm{Th}_d(\tau)$ be the space of $\Lambda(1,d)$-invariant theta functions with characteristics. Then $\mathrm{Th}_d(\tau)$ is a d-dimensional vector space with a basis

$$\vartheta\begin{bmatrix} k/d \\ 0 \end{bmatrix}, \quad 0 \le k \le d-1.$$

By (5) and (7), one has the relations:

$$T_{1/d}\vartheta\begin{bmatrix} k/d \\ 0 \end{bmatrix} = \vartheta\begin{bmatrix} (k+1)/d \\ 0 \end{bmatrix}, \qquad S_1\vartheta\begin{bmatrix} k/d \\ 0 \end{bmatrix} = \mathrm{e}^{2k\pi\mathrm{i}/d}\vartheta\begin{bmatrix} k/d \\ 0 \end{bmatrix}.$$

By (2), (3) and (8) , the action of $T_{1/d}$, S_1 on $\mathrm{Th}_d(\tau)$, together with the involution

$$\vartheta\begin{bmatrix} k/d \\ 0 \end{bmatrix}(z) \longmapsto \vartheta\begin{bmatrix} k/d \\ 0 \end{bmatrix}(-z),$$

gives rise to the canonical representation of the extended degree d Heisenberg group $\tilde{\mathbb{G}}_d$ via the identification $e_k = \vartheta\left[\begin{smallmatrix} k/d \\ 0 \end{smallmatrix}\right]$.

3. ODEs for mirror maps of elliptic curves

In this section, we shall characterize the differential operators in Table 1 through J-function with an emphasis on integral $\mathbf{q}$-expansion property of the variable z . It is known that an elliptic curve can be represented in the Weierstrass form:

$$y^2 = 4x^3 - g_2x - g_3, \quad (x,y) \in \mathbb{C}^2, \tag{12}$$

with the parameter $[g_2, g_3] \in \mathbb{P}^1_{(2,3)}$, equivalently the variable J $(:= g_2^3/(g_2^3 - 27g_3^2) \in \mathbb{C} \cup \{\infty\})$. The periods of elliptic curves satisfy the Picard–Fuchs equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}J^2} + \frac{1}{J}\frac{\mathrm{d}y}{\mathrm{d}J} + \frac{31J-4}{144J^2(1-J)^2}y = 0.$$

A ratio of two periods gives the variable τ of $\mathbb{H}$, which, as a function of J, satisfies the Schwarzian equation:

$$\{\tau, J\} = \frac{3}{8(1-J)^2} + \frac{4}{9J^2} + \frac{23}{72J(1-J)},$$

here the Schwarzian derivative is defined by $\{y, x\} = y'''/y' - \frac{3}{2}(y''/y')^2$. The inverse function $J(\tau)$ of $\tau(J)$ is a modular function [7,14] with $g_2(\tau)$, $g_3(\tau)$ expressed by Eisenstein series. Hence $1728J(\tau)$ admits an integral $\mathbf{q}$-series:

$$1728J(\tau) = \mathbf{q}^{-1} + 744 + 196884\mathbf{q} + 21493760\mathbf{q}^2 + \cdots, \qquad \mathbf{q} = \mathrm{e}^{2\pi \mathrm{i}\tau}.$$

In this section, we shall discuss the integral property of the $\mathbf{q}$-series in (1) and the relation between J and z in Table 1. We state the following simple lemma for later use.

Lemma 1. *Let $w(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $z(\mathbf{q}) = \mathbf{q} + \sum_{m=2}^{\infty} k_m \mathbf{q}^m$ be two formal power series with $a_m \in \mathbb{Z}$ for all $m \geq 2$. Then $w(z(\mathbf{q}))$ has an integral $\mathbf{q}$-expansion if and only if all the k_m's are integers.* □

Consider the following ODEs of Fuchsian type:

$$(\Theta^2 - \rho z(\Theta + \alpha)(\Theta + \beta))y(z) = 0, \qquad \rho, \alpha, \beta \in \mathbb{Q} > 0, \quad \alpha + \beta = 1, \quad \alpha > \beta, \tag{13}$$

where $\Theta = z(\mathrm{d}/\mathrm{d}z)$. Eq. (13) is invariant under the change of variables $z \longmapsto -z + 1/\rho$, and with three regular singular points $z = 0, 1/\rho, \infty$. By the change of variables, $x = \rho z$, (13) becomes the hypergeometric equation:

$$x(1-x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1-2x)\frac{\mathrm{d}y}{\mathrm{d}x} - \alpha\beta y = 0, \tag{14}$$

whose fundamental solutions at $x = 0$ consist of a hypergeometric series and another solution with logarithmic term:

$$y_1(x) = F(\alpha, \beta; 1; x), \qquad y_2(x) = \log(x)F(\alpha, \beta; 1; x) + \sum_{n=1}^{\infty} a_n x^n.$$

The local system for Eq. (14) is described by analytic continuations of $y_1(x)$ and $y_2(x)$, or of another pair of fundamental solutions, $ay_1(x) + by_2(x)$ and $cy_1(x) + dy_2(x)$. The ratio

$$t(x) = \frac{ay_1(x) + by_2(x)}{ay_1(x) + by_2(x)}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}),$$

is called a Schwarz (triangle) function, which satisfies the Schwarzian differential equation:

$$\{t, x\} = 2Q(x) \quad \text{with } Q(x) = \frac{1 - 4\alpha\beta x(1-x)}{4x^2(1-x)^2}. \tag{15}$$

All the Schwarz functions are equivalent under $SL_2(\mathbb{C})$-action on the value t. Now the solutions of Eq. (13) are expressed by Riemann P-function

$$P\begin{Bmatrix} 0 & 1/\rho & \infty & \\ 0 & 0 & \alpha & ; z \\ 0 & 0 & \beta & \end{Bmatrix} = Af_1(z) + Bf_2(z), \qquad A, B \in \mathbb{C},$$

with

$$f_1(z) = y_1(\rho z) = F(\alpha, \beta; 1; \rho z),$$
$$f_2(z) = y_2(\rho z) - \log(\rho) y_1(\rho z) = \log(z) f_1(z) + \sum_{n=1}^{\infty} d_n z^n.$$

The ratio

$$\mathbf{t}(z) = \frac{f_2(z)}{2\pi \mathrm{i} f_1(z)} \tag{16}$$

forms a uniformizing coordinate of the punctured disc at $z = 0$, which is characterized by the solution of the equation

$$\{t, z\} = 2\rho^2 Q(\rho z) \tag{17}$$

with the conditions:

$$\lim_{z\to 0} \mathbf{t}(z) = \infty, \qquad \lim_{\theta \to 2\pi^-} \mathbf{t}(\mathrm{e}^{\mathrm{i}\theta} z) = \mathbf{t}(z) + 1, \qquad \lim_{z\to 0} \frac{\mathbf{q}}{z} = 1 \quad (\mathbf{q} := \mathrm{e}^{2\pi \mathrm{i} \mathbf{t}}). \tag{18}$$

We have a local isomorphism between the z and $\mathbf{q}$-planes with the relation

$$z = \mathbf{q} + \sum_{n\geq 2} k_n \mathbf{q}^n, \qquad k_n \in \mathbb{C}. \tag{19}$$

The characterization of the equations of type (13) with integral coefficients for its associated series (19), i.e. $k_n \in \mathbb{Z}$ for all n, will be our main concern in what follows.

The analytical continuation of $\mathbf{t}(z)$ gives rise to a Riemann surface $\Re$ which spreads over the $\mathbf{t}$-plane and infinitely covers z-plane outside $\{0, 1/\rho, \infty\}$. We have the following relations between Riemann surfaces:

$$\mathbb{P}^1 - \{0, 1/\rho, \infty\} \xleftarrow{z} \Re \xrightarrow{\mathbf{t}} \mathbf{t}(\Re) \subset \mathbb{P}^1.$$

One can extend $\Re$ to a Riemann surface $\overline{\Re}$ over the ∞-value of z with an extended diagram

$$\mathbb{P}^1 - \{0, 1/\rho\} \xleftarrow{z} \overline{\Re} \xrightarrow{\mathbf{t}} \mathbf{t}(\overline{\Re}) \subset \mathbb{P}^1$$

such that near an element of $z^{-1}(\infty)$, the local description of the above diagram is equivalent to the following one:

$$\{|1/z| < \delta'\} \longleftarrow \{|s| < \epsilon\} \longrightarrow \{|\tilde{t}| < \delta\}, \qquad 1/z = s^k \longleftarrow \quad s \quad \longrightarrow \tilde{t} = s^l,$$

where k, l are two relatively prime positive integers with $\alpha - \beta = l/k$ (hence $k \geq 2$), and $\tilde{t}$ is a local coordinate of $\mathbf{t}$-plane centered at $\mathbf{t}(\infty)$. Similarly the (multi-valued) function $\tau(J)$, $J \neq 0, 1$, defines a Riemann surface $\Re_J$ with its partial compactification $\overline{\Re_J}$:

$$\mathbb{C} \xleftarrow{J} \overline{\Re_J} \overset{\tau}{\simeq} \mathbb{H} \subset \mathbb{P}^1.$$

We introduce a notion relating $\mathbf{t}$ and τ for our later use.

Definition 2. The Schwarz triangle function $\mathbf{t}(z)$ (16) for Eq. (13) is related to J-function if for some morphisms Ψ and Φ, the following diagram commutes:

$$\begin{array}{ccccccc} \mathbb{P}^1 \supset \mathbb{P}^1 - \{0, 1/\rho\} & \xleftarrow{z} & \overline{\mathfrak{R}} & \xrightarrow{\mathbf{t}} & \mathbf{t}(\overline{\mathfrak{R}}) & & \\ \Psi\downarrow \qquad \downarrow & & \Phi\downarrow & & \| & & \\ \mathbb{P}^1 \supset \qquad \mathbb{C} & \xleftarrow{J} & \overline{\mathfrak{R}_J} & \overset{\tau}{\simeq} & \mathbb{H} & \subset & \mathbb{P}^1 \end{array} \tag{20}$$

Theorem 1. *All the differential operators of type* (13) *with* $\mathbf{t}(z)$ *related to* J*-function and the integral* $\mathbf{q}$*-series* $z(\mathbf{q})$ *are those listed in Table* 1.

The rest of this section will be mainly devoted to the proof of Theorem 1. We shall regard a coordinate system of $\mathbb{C}$ as the (affine) coordinate of the Riemann sphere $\mathbb{P}^1$ via the identification: $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. As before, by the change of variables $x = \rho z$, the morphism Ψ in (20) induces a rational map

$$\psi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \quad x \longmapsto J = \psi(x) .$$

By examining the behavior over critical values of the function $J(\tau)$, the above morphism ψ satisfies the following conditions:

(i) The critical values of ψ are contained in $\{0, 1, \infty\}$.

(ii) $\psi^{-1}(\infty) = \{0, 1\}$, and the multiplicity of ψ at 0 is equal to 1, i.e. $\mathrm{mult}_\psi(0) = 1$.

(iii) The value $\psi(\infty)$ is equal to 0 or 1. For $x \neq \infty$ and $\psi(x) = 0, 1$, we have $\mathrm{mult}_\psi(x) = 3, 2$ according to $\psi(x) = 0$ or 1, respectively.

Lemma 2. *There are exactly three solutions for the above function* $\psi(x)$*:*

$$\frac{(1+8x)^3}{64x(1-x)^3}, \qquad \frac{(1+3x)^3}{27x(1-x)^2}, \qquad \frac{1}{4x(1-x)} .$$

Proof. Let d be the degree of the map ψ. By the conditions on ψ, d is greater than or equal to 2. Further $d = 2$ if and only if {critical value of ψ} = $\{0, 1\}$, in which case one has $\psi(\infty) = 0$, then it is easy to see $\psi(x) = 1/4x(1-x)$. Now assume $d \geq 3$. By Hurwitz Theorem, $2d - 2 = r_0 + r_1 + r_\infty$ where r_j is the sum of ramification indices of elements in $\psi^{-1}(j)$. By (ii), $r_\infty = d - 2$, hence $d = r_0 + r_1$. Let k be the multiplicity of ψ at $x = \infty$. By (iii),

$$(r_0, r_1) = \left(2\frac{d-k}{3} + k - 1\,,\ \frac{d}{2}\right), \left(2\frac{d}{3}\,,\ \frac{d-k}{2} + k - 1\right)$$

according to $\psi(\infty) = 0$ or 1, respectively. This implies either $(d, k) = (4, 1)$, $\psi(x) = a(x+b)^3/x(1-x)^3$ or $(d, k) = (3, 1)$, $\psi(x) = (x+b)^3/x(1-x)^2$ for some complex numbers $a \neq 0$, $b \neq 0, -1$. For $d = 3$, there is only one critical point not in $\{0, 1, \infty, -b\}$, which is given by $x = b/(3b+2)$. We have $\psi(b/(3b+2)) = 1$, hence $27b^3 + 27b^2 - 4 = 0$ which implies $b = \frac{1}{3}$. Therefore $\psi(x) = (1+3x)^3/27x(1-x)^2$. For $d = 4$, there are exactly two critical points x_1, x_2 not in $\{0, 1, \infty, -b\}$, and $\mathrm{mult}_\psi(x_i) = 2$, $\psi(x_i) = 1$ for $i = 1, 2$. By the expression of $\psi(x)$, one can easily see that x_i's are the solutions of the relation, $x^2+(2+4b)x-b=0$, with the non-zero discriminant $4(4b+1)(b+1)$. Then x_1, x_2 satisfy the following relations for $x = x_i$,

$$x+b=\frac{3x(1-x)}{1-4x},$$
$$\psi(x)=\frac{27ax^2}{(1-4x)^3}=\frac{27a((-2-4b)x+b)}{(-108-256b-64(2+4b)^2)x+1+48b+64(2+4b)b}.$$

Since $\psi(x_1)=\psi(x_2)=1$ and $x_1 \neq x_2$, this implies the vanishing of the determinant of coefficients in the above expression for $\psi(x)$, hence $(8b-1)(4b+1)=0$ and $b=\frac{1}{8}$. Therefore $\psi(x)=(1+8x)^3/64x(1-x)^3$. □

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. By Lemma 2, the map Ψ in (20) and the corresponding value of $\alpha-\beta$ are expressed by

$$1728J = 1728\Psi(z) = \begin{cases} C\dfrac{(1+8\rho z)^3}{z(1-\rho z)^3}, \quad C=\dfrac{1728}{64\rho}, \quad \alpha-\beta=\dfrac{1}{3}, \\ C\dfrac{(1+3\rho z)^3}{z(1-\rho z)^2}, \quad C=\dfrac{1728}{27\rho}, \quad \alpha-\beta=\dfrac{1}{2}, \\ C\dfrac{1}{z(1-\rho z)}, \quad C=\dfrac{1728}{4\rho}, \quad \alpha-\beta=\dfrac{2}{3}. \end{cases}$$

Since both $1728J$ and z have integral $\mathbf{q}$-expansions by Lemma 1, C is equal to 1, hence $\rho = 27, 64, 432$ according to $\alpha-\beta=\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, respectively. The corresponding operators (13) are given in Table 1. Therefore those are the only possible differential operators satisfying the conditions of Theorem 1. Now we are going to show that the function $\mathbf{t}(z)$ associated to an operator in Table 1 does arise from J-function with the corresponding expression of $1728J(z)$ given there, hence implies the integral $\mathbf{q}$-series for $z(\mathbf{q})$ by Lemma 1. Associated to each family of the weighted hypersurfaces in Table 1, there corresponds an algebraic surface, denoted again by the same symbol,

$$P_8:\ x_1^3+x_2^3+x_3^3-sx_1x_2x_3=0,\ ([x_1,x_2,x_3],[1,s]) \in \mathbb{P}^2 \times \mathbb{P}^1,$$
$$X_9:\ x_1^4+x_2^4+x_3^2-sx_1x_2x_3=0,\ ([x_1,x_2,x_3],[1,s]) \in \mathbb{P}^2_{(1,1,2)} \times \mathbb{P}^1,$$
$$J_{10}:\ x_1^6+x_2^3+x_3^2-sx_1x_2x_3=0,\ ([x_1,x_2,x_3],[1,s]) \in \mathbb{P}^2_{(1,2,3)} \times \mathbb{P}^1$$

with the singular set given by $\emptyset$, $\{([0,0,1],[1,\infty])\}$, $\{([0,1,0],[1,\infty]), ([0,0,1],[1,\infty])\}$, respectively. Let S $(= S(P_8), S(X_8), S(J_{10}))$ be the corresponding minimal resolution, which is an elliptic surface over $\mathbb{P}^1$ via the s-projection, $\sigma : S \longrightarrow \mathbb{P}^1$, with $\sigma^{-1}(\infty)$ as a singular fiber of type ${}_1\mathrm{I}_b$ for $b = 3, 4, 6$ according to $S = S(P_8), S(X_9), S(J_{10})$, respectively, here ${}_1\mathrm{I}_b = E_1 + \cdots + E_b$, the union of b rational curves with the only intersections, $E_j \cdot E_{j+1} = 1$ for $1 \le j \le b$ and $E_{b+1} := E_1$. Let $\{\Gamma_1, \Gamma_2\}$ be a canonical basis of $\mathrm{H}_1(\sigma^{-1}(s), \mathbb{Z})$ for $|s| \gg 0$ with Γ_1 the vanishing circle near a double point of $\sigma^{-1}(\infty)$ and Γ_2 the invariant circle near $\sigma^{-1}(\infty)$. The Picard–Lefschetz transformation along $se^{i\theta}$ as θ varies from 0 to -2π leaves Γ_1 invariant and sends Γ_2 to $\Gamma_2 + b\Gamma_1$. Since the morphism, $([x_1, x_2, x_3], [1, s]) \longmapsto ([e^{2\pi i/b}x_1, x_2, x_3], [1, e^{-2\pi i/b}s])$, induces an order b automorphism of the surface S, the period map, $s \longmapsto (\int_{\Gamma_2}\omega_s)/(\int_{\Gamma_1}\omega_s)$ where $\omega_s =$ the holomorphic differential of $\sigma^{-1}(s)$, is determined by the variable $z(:= s^{-b})$. Hence we obtain a (multi-valued) function $\mathbf{t}(z)$ from z-plane to the upper half-plane $\mathbb{H}$, which satisfies the corresponding differential equation in Table 1 (for its derivation , see e.g. [8]). As z varies along the path $ze^{i\theta}$ from $\theta = 0$ to $\theta = 2\pi$, the homology class Γ_1 on a fiber is unchanged, while Γ_2 becomes $\Gamma_2 + \Gamma_1$. Therefore $\mathbf{t}(z)$ satisfies condition (18). The function $\mathbf{t}(z)$ describes the periods of elliptic fibers of the surface S, hence arises from J-function and we have the diagram (20). By Lemma 2 and the possible forms for $1728J(z)$ we have already known, the result follows immediately. This completes the proof of Theorem 1. □

From the relation of variables J and z in Table 1, the map $\mathbf{t}$ in (20) is bijective on the region $\mathrm{Im}(\mathbf{t}) \gg 0$. Note that $\mathbf{t}$ is also bijective on a connected region of $\Re$ over $|z| \gg 0$ for the cases of P_8 and X_9, but not for J_{10}. For J_{10}-family, the Riemann surface $\overline{\Re}$ is not isomorphic to $\mathbb{H}$. However, from the relation of J and z, one can easily obtain the relation of $z(\tau)$ with Eisenstein series. For an elliptic curve in J_{10}-family:

$$(J_{10}) \quad X_s\colon\ x_1^6 + x_2^3 + x_3^2 - sx_1x_2x_3 = 0, \quad [x_1, x_2, x_3] \in \mathbb{P}^2_{(1,2,3)},$$

the substitution,

$$\mathrm{i}y = \frac{2x_3 - sx_2x_1}{x_1^3}, \qquad x = \frac{12x_2 - s^2x_1^2}{12x_1^2}, \tag{21}$$

changes X_s to the Weierstrass form (12) with $g_2 = \frac{1}{12}s^4$, $g_3 = \frac{1}{216}s^6 - 4$. By using the homogeneous coordinates of $\mathbb{P}^2_{(1,2,3)}$, $[y_1, y_2, y_3] = [1, x, \mathrm{i}y]$, the Weierstrass form is now given by

$$y_3^2 + 4y_3^3 - g_3y_1^6 - g_2y_1^4y_2 = 0, \quad [y_1, y_2, y_3] \in \mathbb{P}^2_{(1,2,3)}.$$

Through Weierstrass function presentation of (12) and the formulae in [16], one obtains the theta function expression of the above y_i's:

$$\begin{aligned} y_1 &= 2\vartheta_1(z, \tau)\,, \\ y_2 &= \tfrac{1}{3}[\vartheta_3^4(0, \tau) + \vartheta_3^4(0, \tau)]\vartheta_1^2(z, \tau) + \vartheta_3^2(0, \tau)\vartheta_4^2(0, \tau)\vartheta_2^2(z, \tau), \\ y_3 &= -2\mathrm{i}\vartheta_2^2(0, \tau)\vartheta_3^2(0, \tau)\vartheta_4^2(0, \tau)\vartheta_2(z, \tau)\vartheta_3(z, \tau)\vartheta_4(z, \tau). \end{aligned}$$

In a similar way, one can obtain the theta function representation for J_{10}-family via (21).

4. Theta function parametrization for P_8-family

In this section we describe the elliptic theta function representation of elliptic curves in P_8-family,

$$(P_8) \quad X_s\colon\ f_s(x) = x_1^3 + x_2^3 + x_3^3 - sx_1x_2x_3 = 0, \quad [x_1, x_2, x_3] \in \mathbb{P}^2, \tag{22}$$

and express the moduli parameter z ($:= s^{-3}$) by theta constants. First we note that the fundamental locus of the pencil (22) consists of nine elements:

$$[x_1, x_2, x_3] = [0, -1, \omega^k], [-\omega, 0, -1], [-1, \omega^k, 0] \quad \text{for } 0 \le k \le 2, \quad \omega = e^{2\pi i/3}, \tag{23}$$

and each of them gives rise to a section of the elliptic surface:

$$\sigma : S(P_8) \longrightarrow \mathbb{P}^1 . \tag{24}$$

Let G be the group of linear transformations preserving the polynomial $f_s(x)$ for a generic s. It is easy to see that the generators of the finite group G consist of the following elements:

$$\begin{aligned} C(x_1, x_2, x_3) &= (\omega x_1, \omega x_2, \omega x_3), & R(x_1, x_2, x_3) &= (x_1, \omega x_2, \omega^2 x_3), \\ T(x_1, x_2, x_3) &= (x_2, x_3, x_1), & I(x_1, x_2, x_3) &= (x_1, x_3, x_2). \end{aligned} \tag{25}$$

The diagonal subgroup of G is generated by C and R. By the relation $R \cdot T = C\ (T \cdot R)$, the group G is isomorphic to the extended degree 3 Heisenberg group $\tilde{\mathbb{G}}_3$:

$$G \cap SL_3(\mathbb{C}) = \langle C, R, T \rangle \simeq \mathbb{G}_3, \qquad G = \langle C, R, T \rangle \cdot \langle I \rangle \simeq \tilde{\mathbb{G}}_3 . \tag{26}$$

The action of G on the homogeneous coordinates x_k's is equivalent to the canonical representation of $\tilde{\mathbb{G}}_3$ by identifying x_k with e_k of (2). As a projective transformation group, G acts on $\mathbb{P}^2$ and leaves each X_s invariant. Let r_s, t_s, ι_s be the automorphisms of X_s induced by R, T, I, respectively. Then t_s and r_s generates the group of order 3 translations of X_s, and ι_s is an involution of X_s:

$$\langle r_s, t_s \rangle \simeq \mathbb{Z}_3^2 \simeq \mathbb{G}_3/\text{center}(\mathbb{G}_3), \quad \langle r_s, t_s, \iota_s \rangle \simeq \mathbb{Z}_3^2 \cdot \mathbb{Z}_2 \simeq \tilde{\mathbb{G}}_3/\text{center}(\tilde{\mathbb{G}}_3). \tag{27}$$

The fundamental locus (23) is invariant under ι_s with only one fixed element $[0, -1, 1]$, which induces a section of the elliptic surface (24), denoted by $\rho : \mathbb{P}^1 \longrightarrow S(P_8)$. The translations of ρ by elements of $\langle t_s, r_s \rangle$ are the nine sections induced by (23). Denote $\mathcal{O}_{X_s}(1)$ the restriction of hyperplane bundle on X_s. By (26) and (27), we have a $\tilde{\mathbb{G}}_3$-linearization on the line bundle $\mathcal{O}_{X_s}(1)$ via the linear representation of G. We are going to construct this $\tilde{\mathbb{G}}_3$-linearization from the universal covering space of X_s. In the two-dimensional toric variety $\mathbb{P}^2/\langle R \rangle$, there are three toric divisors in $\mathbb{P}^2/\langle R \rangle$ defined by the zeros of ξ_i, $1 \le i \le 3$, where ξ_i are sections on $\mathbb{P}^2/\langle R \rangle$ with $p^*(\xi) = x_i$ under the projection p from $\mathbb{P}^2$ onto $\mathbb{P}^2/\langle R \rangle$. Note that for $i \ne j$, ξ_i and ξ_j are not linearly equivalent, while ξ_i^3's and $\xi_1\xi_2\xi_3$ give rise to sections of the same line bundle. Denote

$$\Xi_s\colon \ \xi_1^3 + \xi_2^3 + \xi_3^3 - s\xi_1\xi_2\xi_3 = 0 \quad \text{in } \mathbb{P}^2/\langle R \rangle. \tag{28}$$

The restriction of the projection p defines a three-fold cover of elliptic curves, $p_s : X_s \longrightarrow \Xi_s$, and the automorphisms ι_s, t_s of X_s induce the involution $\iota_{s,0}$ and an order 3 translation $t_{s,0}$ of Ξ_s, respectively. The restriction of p_s defines a one–one correspondence between the fixed points ι_s and $\iota_{s,0}$, $X_s^{\iota_s} \simeq \Xi_s^{\iota_{s,0}}$, under which $\rho(s)$ corresponds to an element in $\Xi_s^{\iota_{s,0}}$, denoted by $\rho(s)_0 := p_s(\rho(s))$ (= zero(ξ_1)). One may regard Ξ_s as a one-dimensional torus $\mathbb{E}_{\tau,1}$ for some $\tau \in \mathbb{H}$ such that

$$\begin{aligned} \iota_{s,0} : \Xi_s \longrightarrow \Xi_s &\Longleftrightarrow \iota : \mathbb{E}_{\tau,1} \longrightarrow \mathbb{E}_{\tau,1}, \quad [z] \longmapsto [-z] \\ t_{s,0} : \Xi_s \longrightarrow \Xi_s &\Longleftrightarrow t : \mathbb{E}_{\tau,1} \longrightarrow \mathbb{E}_{\tau,1}, \quad [z] \longmapsto [z+c] \ \text{ for } [c] \in \mathbb{E}_{\tau,1}(3), \\ \rho(s)_0 \in \Xi_s &\Longleftrightarrow e \in \mathbb{E}_{\tau,1}(2). \end{aligned} \tag{29}$$

By the following lemma, the above data indeed determine the algebraic form (28).

Lemma 3. *Let $\mathbb{E}$ be an elliptic curve with an involution ι and an order 3 translation t. Let e be an element of $\mathbb{E}$ fixed by ι. Then:*

(i) *We have the linearly equivalent relations of divisors: $e + t(e) + t^2(e) \sim 3e \sim 3t(e) \sim 3t^2(e)$.*

(ii) *There exist non-trivial sections f_i in $\Gamma(\mathbb{E}, \mathcal{O}(t^{i-1}(e)))$, $1 \le i \le 3$, such that the equality, $f_1^3 + f_2^3 + f_3^3 = sf_1f_2f_3$, holds in $\Gamma(\mathbb{E}, \mathcal{O}(3e))$ for some $s \in \mathbb{C} - \{0\}$.*

Proof. Since $\iota(t(e)) = (\iota t \iota)(e) = t^2(e)$, we have $2e \sim t(e) + t^2(e)$. Hence $3e \sim e + t(e) + t^2(e)$. Applying t and t^2 to this relation, we obtain (i). For $1 \le i \le 3$, let f_i be a non-trivial element in $\Gamma(\mathbb{E}, \mathcal{O}(t^{i-1}(e)))$. Since $\Gamma(\mathbb{E}, \mathcal{O}(3e))$ is a three-dimensional vector space with $\{f_i^3\}_{i=1}^3$ as a basis, the relation $f_1f_2f_3 = \beta_1 f_1^3 + \beta_2 f_2^3 + \beta_3 f_3^3$ holds for some complex numbers β_i. As e, $t(e)$ and $t^2(e)$ are three distinct elements in $\mathbb{E}$, one concludes $\beta_i \neq 0$ for all i. Replacing f_i by $\beta_i^{1/3} f_i$, we obtain (ii). □

The above f_i's can be described by theta functions with characteristics on an elliptic curve $\mathbb{E}_{\tau,1}$. One such relation is given as follows.

Lemma 4. *In $\mathbb{E}_{\tau,1}$ ($\tau \in \mathbb{H}$), consider the elements $p_1 = [\frac{1}{2}\tau + \frac{1}{2}]$, $p_2 = [\frac{1}{6}\tau + \frac{1}{2}]$, $p_3 = [\frac{5}{6}\tau + \frac{1}{2}]$. Let*

$$\xi_1 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau), \qquad \xi_2 = \vartheta \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} (z, \tau), \qquad \xi_3 = \vartheta \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} (z, \tau).$$

Then $\xi_1^3, \xi_2^3, \xi_3^3, \xi_1\xi_2\xi_3$ are sections in $\Gamma(\mathbb{E}_{\tau,1}, \mathcal{O}(\sum_{i=1}^3 p_i))$ with the following identity:

$$\xi_1^3 + \xi_2^3 + \xi_3^3 = s\xi_1\xi_2\xi_3 \tag{30}$$

with s given by

$$s^{-1} = \frac{q^{5/9}\vartheta(0,\tau)\vartheta(\tau/3,\tau)\vartheta(2\tau/3,\tau)}{\vartheta(0,\tau)^3 + q^{1/3}\vartheta(\tau/3,\tau)^3 + q^{4/3}\vartheta(2\tau/3,\tau)^3}.$$

Proof. By (6), zero($\xi_1\xi_2\xi_3$) $= p_1 + p_2 + p_3$, and $\xi_1^3, \xi_2^3, \xi_3^3, \xi_1\xi_2\xi_3$ are functions with the same quasi-periodicity condition, hence sections in $\Gamma(\mathbb{E}_{\tau,1}, \mathcal{O}(\sum_{i=1}^3 p_i))$. Now the linear dependence of $\xi_1^3 + \xi_2^3 + \xi_3^3$ and $\xi_1\xi_2\xi_3$ is equivalent to zero($\xi_1^3 + \xi_2^3 + \xi_3^3$) $= p_1 + p_2 + p_3$, which will follow from $(\xi_2^3 + \xi_3^3)(p_1) = (\xi_1^3 + \xi_3^3)(p_2) = (\xi_1^3 + \xi_2^3)(p_3) = 0$. By (6) and (8), we have:

$$\vartheta\begin{bmatrix}2/3\\0\end{bmatrix}(\tfrac{1}{2}(\tau+1),\tau) = -\mathrm{e}^{2\pi\mathrm{i}/3}\vartheta\begin{bmatrix}1/3\\0\end{bmatrix}(\tfrac{1}{2}(\tau+1),\tau),$$

$$\begin{aligned}\vartheta\begin{bmatrix}2/3\\0\end{bmatrix}(\tfrac{1}{2}(\tau+1)-\tfrac{1}{3}\tau,\tau) &= \mathrm{e}^{2\pi\mathrm{i}(\tau+1)/3}\vartheta(\tfrac{1}{2}(\tau+1)+\tfrac{1}{3}\tau)\\ &= \mathrm{e}^{2\pi\mathrm{i}(\tau+1)/3}\vartheta(-\tfrac{1}{2}(\tau+1)-\tfrac{1}{3}\tau)\\ &= -\mathrm{e}^{2\pi\mathrm{i}/3}\vartheta(\tfrac{1}{2}(\tau+1)-\tfrac{1}{3}\tau),\end{aligned}$$

$$\begin{aligned}\vartheta\begin{bmatrix}1/3\\0\end{bmatrix}(\tfrac{1}{2}(\tau+1)-\tfrac{2}{3}\tau,\tau) &= \mathrm{e}^{\pi\mathrm{i}/3}\vartheta(\tfrac{1}{2}(\tau+1)-\tfrac{1}{3}\tau)\\ &= \mathrm{e}^{\pi\mathrm{i}/3}\vartheta(-\tfrac{1}{2}(\tau+1)+\tfrac{1}{3}\tau)\\ &= \mathrm{e}^{\pi\mathrm{i}/3}\vartheta(\tfrac{1}{2}(\tau+1)-\tfrac{2}{3}\tau).\end{aligned}$$

Therefore we obtain relation (30), whose value at $\mathrm{z}=0$ gives the expression of s . □

Remark 1. By a similar argument, one can also have relation (30) by setting:

$$\begin{aligned}&\xi_1=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{z},\tau), && \xi_2=\vartheta\begin{bmatrix}0\\1/3\end{bmatrix}(\mathrm{z},\tau), && \xi_3=\vartheta\begin{bmatrix}0\\2/3\end{bmatrix}(\mathrm{z},\tau),\\ &\xi_1=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{z},\tau), && \xi_2=\mathrm{e}^{8\pi\mathrm{i}/9}\vartheta\begin{bmatrix}1/3\\1/3\end{bmatrix}(\mathrm{z},\tau), && \xi_3=\mathrm{e}^{8\pi\mathrm{i}/9}\vartheta\begin{bmatrix}2/3\\2/3\end{bmatrix}(\mathrm{z},\tau),\\ &\xi_1=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\mathrm{z},\tau), && \xi_2=\mathrm{e}^{-2\pi\mathrm{i}/9}\vartheta\begin{bmatrix}1/3\\2/3\end{bmatrix}(\mathrm{z},\tau) && \xi_3=\mathrm{e}^{-2\pi\mathrm{i}/9}\vartheta\begin{bmatrix}2/3\\1/3\end{bmatrix}(\mathrm{z},\tau)\end{aligned} \tag{31}$$

with the corresponding s given by

$$s^{-1}=\frac{\vartheta(0,\tau)\vartheta(1/3,\tau)\vartheta(2/3,\tau)}{\vartheta(0,\tau)^3+\vartheta(1/3,\tau)^3+\vartheta(2/3,\tau)^3},$$

$$s^{-1}=\frac{q^{5/9}\mathrm{e}^{8\pi\mathrm{i}/9}\vartheta(0,\tau)\vartheta((\tau+1)/3,\tau)\vartheta((2\tau+2)/3,\tau)}{\vartheta(0,\tau)^3+q^{1/3}\mathrm{e}^{-2\pi\mathrm{i}/3}\vartheta((\tau+1)/3,\tau)^3+q^{4/3}\mathrm{e}^{-2\pi\mathrm{i}/3}\vartheta((2\tau+2)/3,\tau)^3},$$

$$s^{-1}=\frac{q^{5/9}\mathrm{e}^{4\pi\mathrm{i}/9}\vartheta(0,\tau)\vartheta((\tau+2)/3,\tau)\vartheta((2\tau+1)/3,\tau)}{\vartheta(0,\tau)^3+q^{1/3}\mathrm{e}^{2\pi\mathrm{i}/3}\vartheta((\tau+2)/3,\tau)^3+q^{4/3}\mathrm{e}^{2\pi\mathrm{i}/3}\vartheta((2\tau+1)/3,\tau)^3}.$$

With the elliptic curve Ξ_s identified with $\mathbb{E}_{\tau,1}$ as in (29), one can write $X_s=\mathbb{C}/L$ for an index 3 sublattice L of $\mathbb{Z}\tau+\mathbb{Z}$. With the complex number c in (29), one may assume that the translation on the complex plane $\mathbb{C}$, $\mathrm{z}\longmapsto\mathrm{z}+\mathrm{c}$, induces the order 3 automorphism t_s of X_s. Then one can easily conclude

$$(X_s, \langle t_s\rangle, \langle r_s\rangle) = \begin{cases} (\mathbb{E}_{\tau,3}, \langle [z] \longmapsto [z+\frac{1}{3}\tau]\rangle, \langle [z] \longmapsto [z+1]\rangle) & \text{if } c = \pm\frac{1}{3}\tau, \\ (\mathbb{E}_{\tau+1,3}, \langle [z] \longmapsto [z+\frac{1}{3}(\tau+1)]\rangle, \langle [z] \longmapsto [z+1]\rangle) & \text{if } c = \pm\frac{1}{3}(\tau+1), \\ (\mathbb{E}_{\tau+2,3}, \langle [z] \longmapsto [z+\frac{1}{3}(\tau+2)]\rangle, \langle [z] \longmapsto [z+1]\rangle) & \text{if } c = \pm\frac{1}{3}(\tau+2), \\ (\mathbb{E}_{3\tau,1}, \langle [z] \longmapsto [z+\frac{1}{3}]\rangle, \langle [z] \longmapsto [z+\tau]\rangle) & \text{if } c = \pm\frac{1}{3}. \end{cases} \tag{32}$$

Consider ξ_i, x_i as functions on the universal covering space $\mathbb{C}$ of Ξ_s, and regard the fundamental group of Ξ_s as a subgroup of the Heisenberg group $\mathbb{G}$ which acts on entire functions of $\mathbb{C}$ as in Section 2. By (32) and (25), ξ_1 corresponds to the $\Lambda(1,1)$-entire function on $\mathbb{C}$, hence $\xi_1 = \vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](z,\tau)$. By Lemma 3, renumbering ξ_2 and ξ_3 if necessary, one may represent ξ_i's by theta functions either in Lemma 4 or those in (31). By (25) and (29), one requires $\xi_2(-z) = \xi_3(z)$, hence by (8), $\xi_2(z) = \vartheta\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right](z,\tau)$ or $\vartheta\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right](z,\tau)$. By (25) and (32), $\xi_2(z+1) = \omega\xi_2(z)$, therefore $\xi_2(z) = \vartheta\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right](z,\tau)$. Then the following conclusion follows from Lemma 4.

Theorem 2. *With the coordinates x_i, ξ_i $(1 \le i \le 3)$ of elliptic curves X_s, Ξ_s in P_8-family, p_s the morphism between them, and r_s, t_s, ι_s the automorphisms of X_s as before, for $\tau \in \mathbb{H}$, define $s(\tau)$ by*

$$s(\tau)^{-1} = \frac{q^{5/9}\vartheta(0,\tau)\vartheta(\tau/3,\tau)\vartheta(2\tau/3,\tau)}{\vartheta(0,\tau)^3 + q^{1/3}\vartheta(\tau/3,\tau)^3 + q^{4/3}\vartheta(2\tau/3,\tau)^3}.$$

Then the above data for X_s, Ξ_s have the following realization in complex tori:

$$X_s = \mathbb{E}_{\tau,3}, \qquad \Xi_s = \mathbb{E}_{\tau,1}, \qquad p_s : X_s \longrightarrow \Xi_s, \quad [z] \longmapsto [z],$$

$$\xi_1 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau), \qquad \xi_2 = \vartheta\begin{bmatrix}1/3\\0\end{bmatrix}(z,\tau), \qquad \xi_3 = \vartheta\begin{bmatrix}2/3\\0\end{bmatrix}(z,\tau),$$

$$t_s : \mathbb{E}_{\tau,3} \longrightarrow \mathbb{E}_{\tau,3}, \quad [z] \longmapsto [z+\tfrac{1}{3}\tau],$$
$$r_s : \mathbb{E}_{\tau,3} \longrightarrow \mathbb{E}_{\tau,3}, \quad [z] \longmapsto [z+1],$$
$$\iota_s : \mathbb{E}_{\tau,3} \longrightarrow \mathbb{E}_{\tau,3}, \quad [z] \longmapsto [-z],$$

and the projective representation of $\langle r_s, t_s, \iota_s\rangle$ on x_i's is given by the canonical representation of $\tilde{\mathbb{G}}_3$ on $Th_3(\tau)$. □

By the above expression of $s(\tau)^{-1}$, we now derive the formula of the variable z $(:= s^{-3})$ in terms of $\mathbf{q}$ as follows.

Theorem 3. *The function $z(\mathbf{q})$ for P_8-family of Table 1 is given by*

$$z(\mathbf{q}) = \frac{\mathbf{q}^{5/2}\vartheta(0,3\mathbf{t})^3\vartheta(\mathbf{t},3\mathbf{t})^3\vartheta(2\mathbf{t},3\mathbf{t})^3}{(\vartheta(0,3\mathbf{t})^3 + \mathbf{q}^{1/2}\vartheta(\mathbf{t},3\mathbf{t})^3 + \mathbf{q}^2\vartheta(2\mathbf{t},3\mathbf{t})^3)^3}, \qquad \mathbf{q} = e^{2\pi i\mathbf{t}},$$

and it has the integral $\mathbf{q}$-expansion with $z(\mathbf{q})/\mathbf{q}$ tending to 1 as $\mathbf{q} \longrightarrow 0$.

Proof. By Theorem 2, for $s = s(\tau)$, we have $X_s = \mathbb{E}_{\tau,3} \simeq \mathbb{E}_{\mathbf{t},1}$ and $\mathbf{t} = \frac{1}{3}\tau$, then one obtains the above expression of $z(\mathbf{q})$. By the infinite q-product representation of the theta function, the ratio $z(\mathbf{q})/\mathbf{q}$ tends to 1 as $\mathbf{q} \longrightarrow 0$, and $z(\mathbf{q})$ is an integral power series of $\sqrt{\mathbf{q}}$. The variable $\mathbf{t}$ is obtained as the ratio of two periods of the holomorphic differential of X_s. As X_s is isomorphic to $X_{\omega s}$, $\mathbf{t}$ can be considered as a multi-valued function of z. Since the periods satisfy Eq. (13) for $\rho = 27, \alpha = \frac{2}{3}, \beta = \frac{1}{3}$, $\mathbf{t}$ is a solution of the corresponding Schwarzian equation (17). It can be shown that $\mathbf{t}(z)$ satisfies condition (18), hence $\mathbf{t}$ is the variable in Section 3. Since z is a function of $\mathbf{q}$, this implies $z(\mathbf{q})$ is a power series of $\mathbf{q}$ with the integral expansion in (1). □

Remark 2. Both the numerator and denominator in the expression of $z(\mathbf{q})$ are integral power series of $\sqrt{\mathbf{q}}$, but not in $\mathbf{q}$, even though their ratio does give an integral $\mathbf{q}$-expansion for z. Also the surface $S(P_8)$ of (24) is the universal family of $(\mathbb{E}, \mathbb{E}(3))$ for 1-torus $\mathbb{E}$ with 3-torsion $\mathbb{E}(3)$.

5. Elliptic curves in Ising model

We now start to investigate the relation between the constrained polynomials of X_9-family and the Boltzmann weights in Ising model. Let us first recall the Jacobi elliptic function parametrization in Ising model. This theory has been extensively discussed. See, e.g. [2,3,15]. Here we adopt the formulation in chiral Potts N-state model [4,13], even though the prime interests of which were on hyperelliptic curves for $N \geq 3$, however the parametrization works also for $N = 2$, i.e. the case of Ising model.

Let W be a one-dimensional torus $\mathbb{E}$ $(= \mathbb{C}/\text{lattice})$, and define the automorphisms θ, σ of W by

$$\theta : [\mathrm{z}] \longmapsto [-\mathrm{z}], \qquad \sigma : [\mathrm{z}] \longmapsto [-\mathrm{z} + \mathrm{z}_0] \quad \text{for some } [\mathrm{z}_0] \in \mathbb{E}(2)\,.$$

One can present W as a plane curve through the following commutative diagram:

$$\begin{array}{ccccc} & W & \xrightarrow{\Psi} & \mathbb{P}^1 & = W/\langle\theta\rangle \\ & \downarrow \Pi & & \downarrow \pi & \\ W/\langle\sigma\rangle = & \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 & = W/\langle\theta,\sigma\rangle\,, \end{array} \tag{33}$$

where Ψ, ψ, Π, π are natural projections. For some suitable coordinates of $\mathbb{P}^1$, ψ, π have the expressions:

$$\psi(t) = t^2, \quad \pi(\lambda) = \frac{(1 - k'\lambda)(1 - k'\lambda^{-1})}{k^2}, \qquad t, \lambda \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\},$$

where $k', k \in \mathbb{C} - \{0, \pm 1\}$ with the relation $k^2 + k'^2 = 1$. Then W is isomorphic to the algebraic curve:

$$W_{k'}:\ t^2 = \frac{(1 - k'\lambda)(1 - k'\lambda^{-1})}{k^2}, \quad (t, \lambda) \in \mathbb{C}^2, \tag{34}$$

which is birationally equivalent to the plane curve:

$$w^2 = (t^2 - (1-k')/(1+k'))(t^2 - (1+k')/(1-k')), \quad (t, w) \in \mathbb{C}^2, \tag{35}$$

via the transformations $w = (k'/k^2)(\lambda - 1/\lambda)$, $\lambda = (1/2k')(k^2(w - t^2) + k'^2 + 1)$. In terms of the coordinates (t, λ), $\Psi, \Pi, \theta, \sigma$ are defined by $\Psi(t, \lambda) = \lambda$, $\Pi(t, \lambda) = t$, $\theta(t, \lambda) = (-t, \lambda)$, $\sigma(t, \lambda) = (t, \lambda^{-1})$ and

$$\begin{aligned} &\text{branched points of } \Psi\colon p = (\infty, 0),\ p' = (\infty, \infty),\ q = (0, k'),\ q' = (0, k'^{-1}), \\ &\text{branched points of } \Pi\colon b_\pm = (\pm\sqrt{(1+k')/(1-k')}, 1), \\ &\qquad\qquad b'_\pm = (\pm\sqrt{(1-k')/(1+k')}, -1)\,. \end{aligned} \tag{36}$$

With $\mathbb{P}^2_{(1,1,2)}$ as a compactification of $\mathbb{C}^2$ via the identification:

$$[1, t, \mathrm{i}w] = [y_1, y_2, y_3] \in \mathbb{P}^2_{(1,1,2)}, \tag{37}$$

Eq. (35) can be rewritten as

$$\begin{aligned} W_{k'} \simeq Y_\epsilon &: y_1^4 + y_2^4 + y_3^2 - \epsilon y_1^2 y_2^2 = 0, \\ &[y_1, y_2, y_3] \in \mathbb{P}^2_{(1,1,2)}, \quad (\epsilon = 2(1+k'^2)/(1-k'^2)). \end{aligned} \tag{38}$$

Now the branched points of Ψ are given by $\{p, p'\} = \{y_1 = 0\}$ and $\{q, q'\} = \{y_2 = 0\}$. The Boltzmann weights a, b, c, d in Ising model can be regarded as sections on $W_{k'}$ for line bundles $\mathcal{O}(q), \mathcal{O}(q'), \mathcal{O}(p'), \mathcal{O}(p)$, respectively [13]. Over a four-fold cover $\tilde{W}_{k'}$ of $W_{k'}$, the above a, b, c, d are linearly equivalent and satisfy the quadratic relations which give rise to the equations of $\tilde{W}_{k'}$ in $\mathbb{P}^3$:

$$\tilde{W}_{k'}\colon \begin{cases} a^2 + k'b^2 = kd^2 \\ k'a^2 + b^2 = kc^2 \end{cases} \iff \begin{cases} ka^2 + k'c^2 = d^2 \\ kb^2 + k'd^2 = c^2 \end{cases} \quad \text{for } [a, b, c, d] \in \mathbb{P}^3. \tag{39}$$

It is known that the variables a, b, c, d have the Jacobi elliptic function parametrization [3,15]. Here we follow the formulation in [4,13] by expressing those variables via the prime form $\vartheta_1(\mathrm{z})(= \vartheta_1(\mathrm{z}, \tau))$ of the elliptic curve. By formulae [2] in [13, pp. 632–633], together with (11), a, b, c, d have the following expression:

$$\begin{aligned} a^2 : b^2 : c^2 : d^2 = &-\mathrm{e}^{2\pi \mathrm{i}\mathrm{z}}\vartheta_1(\mathrm{z} + \tfrac{1}{4}(\tau - 1))^2 : -\vartheta_1(\mathrm{z} + \tfrac{1}{4}(-\tau + 1))^2 \\ &: \vartheta_1(\mathrm{z} + \tfrac{1}{4}(-\tau - 1))^2 : \mathrm{e}^{2\pi \mathrm{i}\mathrm{z}}\vartheta_1(\mathrm{z} + \tfrac{1}{4}(\tau + 1))^2 \end{aligned}$$

[2] A constant was missing in the expression of k' in [13, p.632]. The correct formula for k' is as follows:

$$k' = \frac{-\mathrm{e}^{-\pi \mathrm{i}(\rho_1 + \cdots + \rho_g)}\vartheta\left[\begin{smallmatrix}\bar{\delta}\\ \bar{\nu}\end{smallmatrix}\right](\varepsilon, \tau)^N}{\mathrm{i}^g\, \vartheta\left[\begin{smallmatrix}\bar{\delta}\\ \bar{\nu}\end{smallmatrix}\right](\varrho, \tau)^N}.$$

with

$$k' = -\mathrm{i}\frac{\vartheta_2(0,\tau)^2}{\vartheta_4(0,\tau)^2}\left(= \frac{-\mathrm{e}^{-\pi\mathrm{i}\tau/2}\vartheta_1(1/2,\tau)^2}{\mathrm{i}\,\vartheta_1(\tau/2,\tau)^2}\right), \qquad k = \frac{\vartheta_3(0,\tau)^2}{\vartheta_4(0,\tau)^2}. \tag{40}$$

Within constant factors, the variables a, b, c, d are proportional to the four Jacobi functions $\vartheta_2, \vartheta_4, \vartheta_3, \vartheta_1$ with the same argument. In fact by using (4), one has the following expression:

$$a^2 : b^2 : c^2 : d^2 = \mathrm{i}\vartheta_2(\mathrm{z},\tau)^2 : \vartheta_4(\mathrm{z},\tau)^2 : \vartheta_3(\mathrm{z},\tau)^2 : -\mathrm{i}\vartheta_1(\mathrm{z},\tau)^2.$$

Then Eq. (39) is equivalent to the relations in (10) for Jacobi elliptic functions. By Section 3 of [13], the variables λ, t are related to a, b, c, d by $\lambda = d^2/c^2, t = ab/cd$, hence we obtain theta function representations of λ and t:

$$\begin{aligned} \lambda &= \frac{\mathrm{e}^{2\pi\mathrm{i}\mathrm{z}}\vartheta_1(\mathrm{z}+(\tau+1)/4,\tau)^2}{\vartheta_1(\mathrm{z}-(\tau+1)/4,\tau)^2} = \frac{-\mathrm{i}\vartheta_1(\mathrm{z},\tau)^2}{\vartheta_3(\mathrm{z},\tau)^2}, \\ t &= \frac{\vartheta_1(\mathrm{z}+(\tau-1)/4,\tau)\vartheta_1(\mathrm{z}+(-\tau+1)/4,\tau)}{\vartheta_1(\mathrm{z}-(\tau+1)/4,\tau)\vartheta_1(\mathrm{z}+(\tau+1)/4,\tau)} = \frac{-\mathrm{i}\vartheta_2(\mathrm{z},\tau)\vartheta_4(\mathrm{z},\tau)}{\vartheta_3(\mathrm{z},\tau)\vartheta_1(\mathrm{z},\tau)}. \end{aligned} \tag{41}$$

Note that the Picard–Fuchs equation for the Ising family (38) is equivalent to that of X_9-family. In fact, the equation is derived by Dwork–Griffith–Katz reduction method from residuum expression of the period:

$$\hat{\omega}(s_0, s_1, s_2) = \int_\gamma\int_{\Gamma_i} \frac{y_1\,\mathrm{d}y_2\wedge\mathrm{d}y_3 - y_2\,\mathrm{d}y_1\wedge\mathrm{d}y_3 + 2y_3\,\mathrm{d}y_1\wedge\mathrm{d}y_2}{s_1y_1^4 + s_2y_2^4 + y_3^2 - s_0y_1^2y_2^2},$$

where γ is a small circle in $\mathbb{P}^2_{(1,1,2)}$ normal to the elliptic curve , Γ_i are one-circles on the curve. The above integral is also expressed by

$$\hat{\omega}(1,1,\epsilon) = \frac{-1}{2}\int_{\Gamma_i}\frac{\mathrm{d}t}{w}, \qquad \epsilon = 2\frac{1+k'^2}{1-k'^2},$$

where (t, w) are the coordinates of $W_{k'}$ in (35). It is known that $\hat{\omega}(s_0, s_1, s_2)$ satisfies the following equations:

$$\begin{aligned} &\left(s_0\frac{\partial}{\partial s_0} + s_1\frac{\partial}{\partial s_1} + s_2\frac{\partial}{\partial s_2} + \frac{1}{2}\right)\hat{\omega} \\ &= \left(s_1\frac{\partial}{\partial s_1} - s_2\frac{\partial}{\partial s_2}\right)\hat{\omega} = \left(\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2} - \frac{\partial^2}{\partial s_0^2}\right)\hat{\omega} = 0. \end{aligned} \tag{42}$$

By the ansatz $\hat{\omega}(s_0, s_1, s_2) = (1/\sqrt{s_0})\omega(\zeta),\ \zeta := s_1s_2/s_0^2 = 1/\epsilon^2$, Eq. (42) is brought into the form

$$\left[4(4\zeta-1)\left(\zeta\frac{\partial}{\partial\zeta}\right)^2 + 16\zeta^2\frac{\partial}{\partial\zeta} + 3\zeta\right]\omega(\zeta) = 0\,.$$

By the change of coordinates, $z = -\frac{1}{16}\zeta + \frac{1}{64} = -1/16\epsilon^2 + \frac{1}{64}$, the above equation becomes the differential operator in table 1 for X_9.

6. Jacobi elliptic function parametrization of X_9-family

In this section we investigate the X_9-family:

$$(X_9) \qquad X_s:\ f_s(x) = x_1^4 + x_2^4 + x_3^2 - s x_1 x_2 x_3 = 0, \qquad [x_1, x_2, x_3] \in \mathbb{P}^2_{(1,1,2)}.$$

The curve X_s degenerates at $s = \infty, 2\sqrt{2}\mu$ $(\mu^4 = 1)$, and for $s = 2\sqrt{2}\mu$, it becomes the union of two rational curves. The linear transformation group preserving a generic polynomial $f_s(x)$ is generated by C, R, Σ:

$$\begin{aligned} &C(x_1, x_2, x_3) = (\mathrm{i}x_1, \mathrm{i}x_2, -x_3), \qquad R(x_1, x_2, x_3) = (-x_1, x_2, -x_3), \\ &\Sigma(x_1, x_2, x_3) = (x_2, x_1, x_3). \end{aligned} \tag{43}$$

The diagonal subgroup is generated by C and R, which acts on $\mathbb{P}^2_{(1,1,2)}$ as an order 2 group induced by R. Denote the restriction of R, Σ on X_s by $r_s, \tilde{\sigma}_s : X_s \longrightarrow X_s$. Then r_s is an order 2 translation and $\tilde{\sigma}_s$ is an involution of X_s with $r_s\tilde{\sigma}_s = \tilde{\sigma}_s r_s$. The zeros of x_i's in X_s are given by

$$\mathrm{zero}(x_1) = \{[0, 1, \pm 1]\}, \ \mathrm{zero}(x_2) = \{[1, 0, \pm 1]\}, \ \mathrm{zero}(x_3) = \{[1, \eta, 0], \eta^4 = -1\},$$

each of which is stable under the r_s-action. Via the projection $p: \mathbb{P}^2_{(1,1,2)} \longrightarrow \mathbb{P}^2_{(1,1,2)}/\langle R\rangle$, the coordinates x_i's of $\mathbb{P}^2_{(1,1,2)}$ give rise to sections ξ_i's on $\mathbb{P}^2_{(1,1,2)}/\langle R\rangle$ with $p^*(\xi_i) = x_i$. We have a family of elliptic curves in $\mathbb{P}^2_{(1,1,2)}/\langle R\rangle$:

$$\Xi_s: \xi_1^4 + \xi_2^4 + \xi_3^2 - s\xi_1\xi_2\xi_3 = 0, \qquad [\xi_1, \xi_2, \xi_3] \in \mathbb{P}^2_{(1,1,2)}/\langle R\rangle.$$

The restriction of p defines a two-fold cover of elliptic curves: $p_s : X_s \longrightarrow \Xi_s$. Let σ_s be the involution of Ξ_s induced by $\tilde{\sigma}_s$ with $p_s\tilde{\sigma}_s = \sigma_s p_s$, and p, p', p_3, p_4 be the elements in Ξ_s defined by $\mathrm{zero}(\xi_1) = \{p\}$, $\mathrm{zero}(\xi_2) = \{p'\}$ and $\mathrm{zero}(\xi_3) = \{p_3, p_4\}$. We have $2p \sim 2p'$, $p + p' \sim p_3 + p_4$ and $\xi_1^2, \xi_2^2 \in \Gamma(\Xi_s, \mathcal{O}(2p))$, $\xi_1\xi_2$, $\xi_3 \in \Gamma(\Xi_s, \mathcal{O}(p_3 + p_4))$.

The following lemma shows that qualitatively p, p', p_3, p_4 determine the equation of Ξ_s, however as we shall see it later on, much efforts are required in order to obtain the theta function representation of ξ_i's.

Lemma 5. *Let θ be an involution of an elliptic curve $\mathbb{E}$, and p, p' be two elements in $\mathbb{E}$ fixed by θ. Let m be the order 2 translation of $\mathbb{E}$ with $m(p) = p'$. Then:*

(i) *There exist sections $f_1 \in \Gamma(\mathbb{E}, \mathcal{O}(p))$, $f_2 \in \Gamma(\mathbb{E}, \mathcal{O}(p'))$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathbb{E} & \simeq & \mathbb{E} & \xrightarrow{m} & \mathbb{E} \\ \downarrow & & \downarrow \Psi & & \downarrow \Psi \\ \mathbb{E}/\langle\theta\rangle & \simeq & \mathbb{P}^1 & \xrightarrow{m_0} & \mathbb{P}^1 \end{array}$$

where $\Psi(x) = [f_1^2(x), f_2^2(x)]$ and $m_0([\xi, \eta]) = [\eta, \xi]$.

(ii) *Let p_3 be an element of $\mathbb{E}$ with $\Psi(p_3) = [1, \mathrm{i}]$, and $p_4 := m(\theta(p_3))$. Then $p_3 \neq p_4$ and $4p \sim 4p' \sim 2p_3 + 2p_4 \sim p + p' + p_3 + p_4$. For some $s \in \mathbb{C}$ and $f_3 \in \Gamma(\mathbb{E}, \mathcal{O}(p_3 + p_4))$ with $\mathrm{zero}(f_3) = p_3 + p_4$, the equality, $f_1^4 + f_2^4 + f_3^2 = s f_1 f_2 f_3$, holds in $\Gamma(\mathbb{E}, \mathcal{O}(4p))$.*

Proof. We may assume the map, $\Psi : \mathbb{E} \longrightarrow \mathbb{P}^1 = \mathbb{E}/\langle\theta\rangle$, sends p, p' to $[0,1], [1,0]$, respectively. Since m commutes with θ, the order 2 automorphism m_0 of $\mathbb{P}^1$ induced by m interchanges the elements $[0,1]$ and $[1,0]$, hence it can be described by $m_0([\xi,\eta]) = [\eta,\xi]$. Then the sections f_1, f_2 in (i) are easily obtained . For $x, y \in \mathbb{E}$, $y = m(x)$ if and only if $x + p' \sim p + y$. By the definition of p_3 and p_4, we have the linearly equivalent relations, $2p + p' \sim p_3 + \theta(p_3) + p' \sim p + p_3 + p_4$, hence $p + p' \sim p_3 + p_4$. Then the equivalent relations in (ii) follow immediately. By $\Psi(\theta(p_3)) = \Psi(p_3)$, we have $\Psi(p_4) = m_0(\Psi(\theta(p_3))) = m_0([1,\mathrm{i}]) = [1,-\mathrm{i}]$. Therefore $p_3 \neq p_4$, and $f_1^4(p_j) + f_2^4(p_j) = 0$ for $j = 3, 4$. Let f_3 be a section in $\Gamma(\mathbb{E}, \mathcal{O}(p_3 + p_4))$ with $\mathrm{zero}(f_3) = p_3 + p_4$ and $f_2^4(p) + f_3^2(p) = 0$. The sections $f_1^4 + f_2^4 + f_3^2$, $f_1 f_2 f_3$ in $\Gamma(\mathbb{E}, \mathcal{O}(4p))$ both vanish at p_3, p_4, p, hence they are proportional by a non-zero constant. Therefore we obtain (ii). □

Consider the birational map

$$\phi : \mathbb{P}^2_{(1,1,2)} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \qquad [x_1, x_2, x_3] \longmapsto ([x_1, x_2], [x_3, x_1 x_2]),$$

whose fundamental locus consists of three elements, defined by two of coordinates x_i's being zero. With the automorphism R of $\mathbb{P}^2_{(1,1,2)}$ in (43) and $\tilde{R}$ of $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\tilde{R} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \qquad ([y_1, y_2], [y_3, y_4]) \longmapsto ([y_1, -y_2], [y_3, y_4]),$$

we have $\tilde{R}\phi = \phi R$. Hence ϕ induces a birational morphism between $\mathbb{P}^2_{(1,1,2)}/\langle R\rangle$ and $\mathbb{P}^1 \times \mathbb{P}^1$, $[\xi_1, \xi_2, \xi_3] \longmapsto ([\xi_1^2, \xi_2^2], [\xi_3, \xi_1\xi_2])$, which embeds Ξ_s into $\mathbb{P}^1 \times \mathbb{P}^1$. With the coordinates of Ξ_s in $\mathbb{P}^1 \times \mathbb{P}^1$,

$$\begin{aligned} \Psi : \Xi_s &\longrightarrow \mathbb{P}^1, \quad x \longmapsto [\xi_1^2(x), \xi_2^2(x)], \quad \lambda(x) := (\xi_1^2/\xi_2^2)(x), \\ \Pi' : \Xi_s &\longrightarrow \mathbb{P}^1, \quad x \longmapsto [\xi_3(x), \xi_1\xi_2(x)], \quad u(x) := (\xi_3/\xi_1\xi_2)(x), \end{aligned} \tag{44}$$

Ξ_s is birational to the curve:

$$u^2 - su = -(\lambda + 1/\lambda), \quad (u, \lambda) \in \mathbb{C}^2 .$$

By $\Pi'\sigma_s = \Pi'$ and $\Psi(\sigma_s(\xi)) = \Psi(\xi)^{-1}$ for $\xi \in \Xi_s$, the morphism Π' is equivalent to the projection, $\Xi_s \longrightarrow \Xi_s/\langle\sigma_s\rangle$, and there is an involution θ_s of Ξ_s such that $\Psi\theta_s = \Psi$, $\theta_s\sigma_s = \sigma_s\theta_s$. The branched data of Ψ and Π' are given by

branched points of Ψ: $\quad (u, \lambda) = (\frac{1}{2}s, k'), \ (\frac{1}{2}s, 1/k'), \ (\infty, 0), \ (\infty, \infty)$,

branched points of Π': $\quad (u, \lambda) = (\frac{1}{2}(s \pm \sqrt{s^2 - 8}), 1) \ , \ (\frac{1}{2}(s \pm \sqrt{s^2 + 8}), -1)$

with $k' + 1/k' = \frac{1}{4}s^2$. By changing the variable of $\mathbb{P}^1$ from u to t via

$$t = \frac{2}{\sqrt[4]{s^4 - 64}}(u - \tfrac{1}{2}s) \ \left(= \sqrt{(k'/k^2)}[u - (k' + 1/k')^{1/2}]\right), \tag{45}$$

and defining the morphism

$$\Pi : \Xi_s \longrightarrow \mathbb{P}^1, \qquad \Pi(x) := t(\Pi'(x)), \tag{46}$$

the branched locus Π becomes $t = \pm\sqrt{(1-k')/(1+k')}, \pm\sqrt{(1+k')/(1-k')}$. With Ψ in (44) and Π in (46), one may identify Ξ_s with the curve $W_{k'}$ in (34), which is the same as Y_ϵ in (38). By (44), (45) and (37), the following elliptic curves are isomorphic:

$$\Xi_s \simeq W_{k'} \simeq Y_\epsilon, \qquad [\xi_1, \xi_2, \xi_3] \longleftrightarrow (t, \lambda) \longleftrightarrow [y_1, y_2, y_3], \tag{47}$$

with the correspondences:

$$(t, \lambda) = \left(\frac{2\xi_3 - s\xi_1\xi_2}{\sqrt[4]{s^4 - 64}\xi_1\xi_2}, \frac{\xi_1^2}{\xi_2^2} \right),$$

$$[y_1, y_2, y_3] = [\tfrac{1}{2}\sqrt{s^4 - 64}\xi_1\xi_2, \ \xi_3 - \tfrac{1}{2}s\xi_1\xi_2, \ \mathrm{i}(\xi_1^4 - \xi_2^4)],$$

where parameters s, k' and ϵ are related by $\frac{1}{4}s^2 = k' + 1/k' = 2\epsilon/\sqrt{\epsilon - 4}$. Note that for the X_9-family, $X_s \simeq X_{s_1}$ if and only if $s^4 = s_1^4$, and for the Ising family $W_{k'} \simeq W_{k'_1}$ if and only if $k_1'^2 = 1/k'^2$. Hence the variable z,

$$z := s^{-4} = \frac{-1}{16\epsilon^2} + \frac{1}{64} = \frac{k'^2}{16(1 + k'^2)^2}, \tag{48}$$

is the moduli parameter of isomorphic classes of the elliptic curves either in X_9-family, or in Ising family. According to the discussion in Section 5, one may identify $W_{k'}$ with $\mathbb{E}_{\tau,1}$ where k', τ satisfy relation (40). Using (11), we have

$$z(\tau) = \frac{-\vartheta_2(0,\tau)^4\vartheta_4(0,\tau)^4}{16(\vartheta_3(0,\tau)^8 - 4\vartheta_2(0,\tau)^4\vartheta_4(0,\tau)^4)}, \tag{49}$$

hence

$$s(\tau) = 2\mathrm{e}^{3\pi\mathrm{i}/4}\frac{\sqrt{\vartheta_2(0,\tau)^4 - \vartheta_4(0,\tau)^4}}{\vartheta_2(0,\tau)\vartheta_4(0,\tau)},$$

$$\sqrt[4]{s(\tau)^4 - 64} = 2\mathrm{e}^{\pi\mathrm{i}/4}\frac{\vartheta_3(0,\tau)^2}{\vartheta_2(0,\tau)\vartheta_4(0,\tau)}.$$

Theorem 4. *With the coordinates x_i, ξ_i $(1 \le i \le 3)$ of elliptic curves X_s, Ξ_s in the X_9-family, p_s the morphism between them, and $r_s, \tilde{\sigma}_s$ the automorphisms of X_s as before, for $\tau \in \mathbb{H}$, define $s(\tau)$ by*

$$s(\tau) = 2\mathrm{e}^{3\pi\mathrm{i}/4}\frac{\sqrt{\vartheta_2(0,\tau)^4 - \vartheta_4(0,\tau)^4}}{\vartheta_2(0,\tau)\vartheta_4(0,\tau)}.$$

Then the above data for X_s, Ξ_s have the following realization in complex tori:

$$X_s = \mathbb{E}_{\tau+1,2}, \qquad \Xi_s = \mathbb{E}_{\tau,1}, \qquad p_s : X_s \longrightarrow \Xi_s, \qquad [\mathrm{z}] \longmapsto [\mathrm{z}],$$

$$r_s : \mathbb{E}_{\tau+1,2} \longrightarrow \mathbb{E}_{\tau+1,2}, \qquad [\mathrm{z}] \longmapsto [\mathrm{z}+1],$$

$$\tilde{\sigma}_s : \mathbb{E}_{\tau+1,2} \longrightarrow \mathbb{E}_{\tau+1,2}, \qquad [\mathrm{z}] \longmapsto [-\mathrm{z} + \tfrac{1}{2}(\tau+1)]$$

with ξ_i's given by

$$\xi_1 = \vartheta_1(\mathrm{z}, \tau), \qquad \xi_2 = \mathrm{e}^{\pi \mathrm{i}/4}\vartheta_3(\mathrm{z}, \tau),$$
$$\xi_3 = \frac{\vartheta_3(0,\tau)^2}{\vartheta_2(0,\tau)\vartheta_4(0,\tau)}\vartheta_2(\mathrm{z},\tau)\vartheta_4(\mathrm{z},\tau) - \frac{\sqrt{\vartheta_2(0,\tau)^4 - \vartheta_4(0,\tau)^4}}{\vartheta_2(0,\tau)\vartheta_4(0,\tau)}\vartheta_1(\mathrm{z},\tau)\vartheta_3(\mathrm{z},\tau)$$

and through the Heisenberg group action on entire functions, one has the projective representation of $\langle r_s, \tilde{\sigma}_s\rangle$ *on* x_i*'s:*

$$r_s : (x_1, x_2, x_3) \longmapsto (-x_1, x_2, x_3)$$
$$\tilde{\sigma}_s : (x_1, x_2, x_3) \longmapsto (-\mathrm{e}^{\pi \mathrm{i}/4}x_2, -\mathrm{e}^{\pi \mathrm{i}/4}x_1, \mathrm{e}^{\pi \mathrm{i}/2}x_3).$$

Proof. According to the discussion we have before, with $s = s(\tau)$ one has the identification, $\Xi_s = \mathbb{E}_{\tau,1}$, with the rational functions Ψ, Π in (44) and (46) given by

$$\Psi : \mathbb{E}_{\tau,1} \longrightarrow \mathbb{P}^1 = \mathbb{E}_{\tau,1}/\langle\theta\rangle, \quad \text{where } \theta([\mathrm{z}]) = [-\mathrm{z}],$$
$$\Pi : \mathbb{E}_{\tau,1} \longrightarrow \mathbb{P}^1 = \mathbb{E}_{\tau,1}/\langle\sigma\rangle, \quad \text{where } \sigma([\mathrm{z}]) = [-\mathrm{z} + \tfrac{1}{2}(\tau+1)].$$

Note that the above σ is identified with the automorphism σ_s of Ξ_s, which can be lifted to the automorphism $\tilde{\sigma}_s$ on X_s. Write $X_s = \mathbb{C}/L$ for some index 2 sublattice L of $\mathbb{Z}\tau + \mathbb{Z}$. The morphism p_s is given by the natural projection. On the universal covering space $\mathbb{C}$ of X_s, the affine map, $\mathrm{z} \longmapsto -\mathrm{z} + \frac{1}{2}(\tau+1)$ for $\mathrm{z} \in \mathbb{C}$, induces the order 2 automorphism $\tilde{\sigma}_s$ on X_s, hence the element $\tau + 1$ is in the lattice L. This implies $X_s = \mathbb{E}_{\tau+1,2}$ with $r_s, \tilde{\sigma}_s$ described in the theorem. The Jacobi elliptic function parametrization of ξ_i's now follows from relations (41) and (47), and the actions of $r_s, \tilde{\sigma}_s$ on x_i's are obtained by the formulae in Section 2. □

Now the formula for z $(= s^{-4})$ with $s = s(\tau)$ can be derived from Theorem 4.

Theorem 5. *The function* $z(\mathbf{q})$ *for* X_9*-family of Table* 1 *is given by*

$$z(\mathbf{q}) = \frac{-\vartheta_2(0, 2\mathbf{t}-1)^4\vartheta_4(0, 2\mathbf{t}-1)^4}{16\vartheta_3(0, 2\mathbf{t}-1)^8 - 64\vartheta_2(0, 2\mathbf{t}-1)^4\vartheta_4(0, 2\mathbf{t}-1)^4}$$
$$= \frac{\mathbf{q}\prod_{n=1}^{\infty}(1+\mathbf{q}^n)^8}{\prod_{n=1}^{\infty}(1-\mathbf{q}^{2n-1})^{16} + 64\mathbf{q}\prod_{n=1}^{\infty}(1+\mathbf{q}^n)^8}, \qquad \mathbf{q} = \mathrm{e}^{2\pi \mathrm{i}\mathbf{t}}.$$

As a consequence, $z(\mathbf{q})$ *has an integral* $\mathbf{q}$*-expansion with* $z(\mathbf{q})/\mathbf{q}$ *tending to* 1 *as* $\mathbf{q} \to 0$.

Proof. For $s = s(\tau)$ in Theorem 4, we have the identification: $X_s = \mathbb{E}_{\tau+1,2} \simeq \mathbb{E}_{\mathbf{t},1}, \mathbf{t} = \frac{1}{2}(\tau+1)$. With the theta constant expression (40) for k', relation (48) gives rise the expression of $z(\mathbf{q})$. The same argument as in Theorem 3 shows that $\mathbf{t}$ is the variable described in Section 3. Therefore we have completed the proof of this theorem. □

7. Discussion

In this paper, we have focused on the mathematical structure of "counting" functions $z(\mathbf{q})$, and have developed an analysis of constraints in Table 1 by means of elliptic theta function representations. Now we are going to discuss another aspect, which is possibly of certain physical relevance. Here the explicitly work out example of X_9-family has illustrated its close connection with Ising model in statistical mechanics, where we employ the Jacobi elliptic function parametrization of Ising model to the investigation of X_9-potential. Relation (48) states the parameter z in X_9-family corresponds to the temperature-like parameter k' of Ising model. One interesting point for the derivation of the function $z(\mathbf{q})$ is that on the one hand it is related to Picard–Fuschs equation of the elliptic family, while on the other side with the parametrization for Boltzmann weights of the statistical model, the same result is correctly reproduced. In this setting, the mirror symmetry of X_9-family is connected to the $\mathbb{Z}_2$-cover of elliptic curves in Ising model, which often appears in the theory. Due to the relative simplicity of the models, the quantities involved in our mathematical work usually have interpretations of physical or geometrical meaning, which allows one to compare their essential structures in an explicit way:

N = 2 SUSY LG theory		Two-dimensional exactly solvable model
LG fields	$\longleftrightarrow$	Boltzmann weights
LG superpotential	$\longleftrightarrow$	Yang–Baxter equation
Moduli parameter	$\longleftrightarrow$	Temperature-like parameter
Maximally unipotent area	$\longleftrightarrow$	Low temperature region

Though we do not know now whether other models could be related in a similar manner, it should be interesting to note that in the work carried out in this article, the mathematical structures of corresponding concepts do share a common feature. The theta function parametrization of Ising model we used here has a direct generalization to chiral Potts N-state models. The naive quantitative indications presented by two physical theories, X_9 and Ising models, encourage us to seek a possible link between Calabi–Yau manifolds and chiral Potts models:

$$\begin{array}{ccc} \text{N} = 2 \text{ LG } X_9\text{-theory} & = & \text{Ising model} \\ \downarrow & & \downarrow \\ \text{Kähler manifolds with } c_1 = 0 & \overset{?}{\longleftrightarrow} & \text{chiral Potts } N\text{-state models} \end{array}$$

The connection proposed in the above diagram is vague. Nevertheless some of the symmetries presented in the study of Calabi–Yau spaces resemble those in chiral Potts N-state models. So, we hope some appropriate geometric picture does exist. How to detect this novel phenomena should be of merit for further investigation.

Acknowledgements

The author would like to thank A. Klemm, B.H. Lian, and S.T. Yau for discussions on mirror symmetry, and also S.S. Lin for discussions on two-dimensional solvable statistical models.

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